Ergodic Dynamics in a Natural Threshold System

K. F. Tiampo,[1,*] J. B. Rundle,[2] W. Klein,[3] J. S. Sá Martins,[4] and C. D. Ferguson[5]
[1]*CIRES, University of Colorado, Boulder, Colorado 80309, USA*
[2]*Center for Computational Science and Engineering, University of California, Davis, California 95616, USA*
[3]*Department of Physics, Boston University, Boston, Massachusetts 02215, USA and Center for Nonlinear Science, Los Alamos, New Mexico 87545, USA*
[4]*Instituto de Fisica, Universidade Federal Fluminense, Niteroi 24210-340 Brazil*
[5]*Center for Nonproliferation Studies, Monterey Institute of International Studies, Washington, D.C. 20036, USA*
(Received 10 March 2003; published 5 December 2003)

Numerical simulations suggest that certain driven, dissipative mean-field threshold systems, including earthquake models, can be characterized by statistical properties often associated with ergodic dynamics, in the same sense as stochastic Brownian motion. We applied a fluctuation metric proposed by Thirumalai and Mountain [Phys. Rev. E **47**, 479 (1993)] for statistically stationary systems and find that the natural earthquake fault system in California demonstrates similar ergodic dynamics.

DOI: 10.1103/PhysRevLett.91.238501 PACS numbers: 91.30.Px, 05.40.Fb, 05.70.Jk, 89.75.Fb

Driven threshold systems represent some of the most important nonlinear, self-organizing systems in nature, and include neural nets, magnetic depinning transitions in superconductors, domain rearrangements in flowing foams, charge density waves in semiconductors, earthquake fault networks, and the World Wide Web, as well as many political, social, and ecological systems [1–7]. All of these systems have dynamics that are strongly correlated in space and time, and all typically display a multiplicity of spatial and temporal scales. If the range of interactions between elements is long and the coupling weak, so that the dynamics can be interpreted as mean field, fluctuations tend to be suppressed and the system may approach a stationary state [8]. Recent work using computer simulations of simplified models of natural dissipative systems suggests that certain equilibriumlike properties may be recovered at the appropriate spatiotemporal scales [1,8,9]. In particular, a number of investigators have shown that models of statistically stationary, driven dissipative mean-field systems demonstrate effectively ergodic dynamics and these model systems reside in a sequence of physical states that are similar to equilibrium, or metastable equilibrium, states [1,10–12]. To date, however, there has been no evidence that the corresponding naturally occurring systems demonstrate the same effectively ergodic, or equilibriumlike, properties.

Summary of our results.—We have applied a fluctuation metric developed by Thirumalai and Mountain to test for the presence of ergodic dynamics in data obtained from (a) simulated earthquakes arising on a typical mean-field slider block model for earthquake dynamics and (b) natural earthquakes observed on the California fault system. Previous work [1,10,13] suggests that the elastic interaction, which is known to be long range, is responsible for inducing a mean-field condition in earthquake fault systems. We find similar results in both cases, providing support for the conclusion that the stationary dynamics of natural earthquake fault systems also represent effectively ergodic, equilibriumlike dynamics.

Threshold systems.—Driven nonlinear threshold systems are composed of interacting spatial networks of cells, each having one or more inputs, an internal state variable $\sigma(t)$ that evolves in time in response to inputs, and one or more outputs. Each cell is connected to other cells by means of a network of interactions, and to an external driving source. Threshold dynamics arise when a cell is subjected to this persistent external forcing, increasing the value of $\sigma(t)$ through time until a predefined failure threshold σ^{F} is reached, where the cell fails, reducing $\sigma(t)$ to a residual value σ^{R}. Thresholds, residual values, internal states, and the resulting dynamics may be modified by the presence of noise and disorder. Mean-field threshold systems arise when the coupling between oscillators is long range but weak, leading to suppression of all but the longest wavelength fluctuations. The system dynamics often result in strong space-time correlations in oscillator firings over many scales [13].

As the interaction length becomes large, a mean-field spinodal appears that is the classical limit of stability of a spatially extended system [14]. Examined in this limit, driven threshold systems possess statistically stationary dynamics and display equilibriumlike behavior when driven at a uniform rate. Following the initial discovery that driven mean-field slider block systems with microscopic noise display equilibrium properties, other studies have confirmed local ergodicity, the existence of Boltzmann fluctuations in these and other mean- or near mean-field systems, and the appearance of an energy landscape, similar to other equilibrium systems [1,9,10,12]. Thus the origin of the physics of scaling, critical phenomena, and nucleation appears to lie, at least in part, in the ergodic properties of these mean-field systems.

Models.—Here we are concerned with driven mean-field threshold systems characterized by Langevin dynamics with additive noise. We consider the slider block model for earthquake faults, a two-dimensional network of blocks arranged in a regular lattice pattern ($d = 2$) sliding on a frictional surface. Each block is connected to q other blocks by means of coupling springs and to a loader plate by means of a separate spring. The persistent loader plate motion raises the stress level on all blocks over time. This particular slider block model has long-range interactions, resulting in mean-field behavior [1,10,14].

A combination of theoretical analysis and numerical simulations established the link between earthquake fault networks and the physics of critical point systems [1,9,10,15,16]. As in other threshold systems, interactions in the natural earthquake system occur along a spatial network of cells, or fault segments, mediated by means of a potential that allows stresses to be redistributed following slip on any particular segment. A persistent driving force, arising from plate tectonic motions, increases stress on the fault segments [1,10,13]. Once the stresses reach a threshold characterizing the limit of stability of the fault, a sudden slip event occurs.

The fundamental elastic interaction between fault patches results in the formation of a mean-field regime for the earthquake system due to the nature of the stress Green's function (proportional to $1/|x - x'|^3$). Long-range interactions lead to an averaging of stress over the system. Longer spatial and temporal wavelengths become increasingly important, and correlation lengths become increasingly larger as they approach a critical point, in association with power law scaling similar to the Gutenberg-Richter relation [1,9,10,13,15,16].

In numerical simulations, these mean-field earthquake networks are nonequilibrium systems that can be treated as equilibrium systems as they settle into a metastable equilibrium state. The time-averaged elastic energy of the system fluctuates around a constant value for some long period of time, punctuated by major events which reorder the system before it settles into another metastable well around a new mean energy state [1,10,13].

The spatial and temporal firing patterns of driven threshold systems are complex and difficult to understand and interpret from a deterministic perspective, as they develop from the obscure underlying parameters and dynamics of a multidimensional nonlinear system [17]. Similarly, there is no means at present to measure the stress and strain at every point in an earthquake fault system, or the constitutive parameters that characterize this heterogeneous medium [18]. However, seismicity, the firing patterns that are the surface proxy for the dynamical state of the underlying fault system, can be located in both space and time with considerable accuracy [19]. Here we compare the ergodic properties of the simulated and natural systems in terms of their energy release. We employ the Thirumalai-Mountain (TM) fluctuation metric in order to make this comparison [10,11,20].

TM fluctuation metric.—The TM metric measures effective ergodicity, or the difference between the time average of a quantity, generally related to the energy E_j at each site, and its ensemble average over the entire system. The fundamental idea is that of statistical symmetry, in which the N particles in the system are statistically identical in terms of their averaged properties—the statistics of one particle are the same as those for the entire system. While, classically, a system is ergodic for infinite averaging times, if the actual measurement time scales are finite, but long, all regions of phase space are sampled with equal likelihood and the system is effectively ergodic [11,20]. Practically this means that over a large enough representative sample in time and space, the spatial and temporal averages are constant. Classical ergodicity is a behavior that is limited to equilibrium states, in which transition probabilities are univarying or follow a definite cycle, and implies stationarity. If such a system is ergodic, it is in metastable equilibrium and can be analyzed as such.

The fluctuation metric $\Omega_e(t)$ is

$$\Omega_e(t) = \frac{1}{N}\sum_{i=1}^{N}[\varepsilon_i(t) - \bar{\varepsilon}(t)]^2, \qquad (1)$$

where $\varepsilon_i(t)$ is the time average of an individual property, $E_i(t)$, and $\bar{\varepsilon}(t)$ is the ensemble average over the entire system. If the system is effectively ergodic at long times, $\Omega_e(t) = \frac{D}{t}$, where D is a diffusion constant that measures the rate of ergodic convergence [11]. The deviation of the time-averaged quantity from its ensemble average [Eq. (1)] is decreasing as a function of time.

In slider block models of earthquake fault networks, as the interaction range increases, the system approaches mean-field limit behavior and can be analyzed using the methods and principles of statistical mechanics. Ferguson *et al.* [10] applied the TM metric to the energy of each block in slider block simulations to show that the system was ergodic at external velocities, V, that approach $V = 0$ (Fig. 1). Data were taken after the slider block system approached a stationary state. Plotted is the inverse of the TM metric, calculated for a loader plate velocity of $V = 0.01$. Note the linear relationship between the inverse TM metric and time, as represented by loader plate update, denoting effective ergodicity as defined above.

Figure 2 shows a similar plot. In this case we plot the inverse TM metric for the numbers of events in a slider block model with a 128×128 member lattice, a loader plate velocity approaching zero, and the addition of a small amount of precursory slip. In this calculation, $E_i(t) \equiv R_i(t)$, number of events greater than a certain magnitude. Number of events is a proxy for energy release. In this case, there is an initial transient phase in addition to the linear sections, where the system exhibits ergodic behavior, punctuated by the occurrence of larger events.

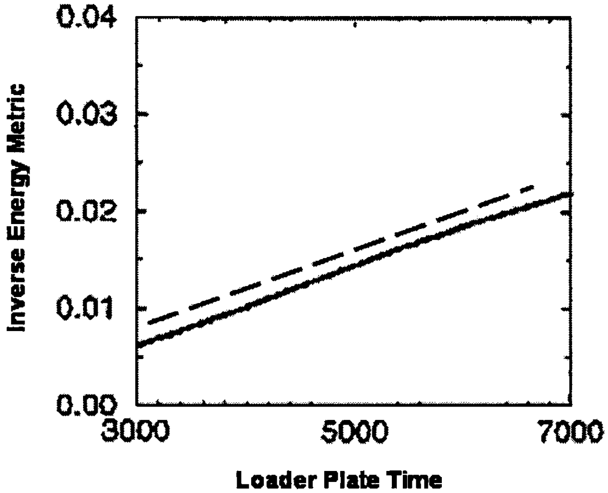

FIG. 1. Inverse of the TM metric vs loader plate update for a 256×256 lattice with closed boundaries, $K_c = K_L = 1$.

While slider block models have been shown to be in metastable equilibrium at the spinodal, and can be analyzed as such [1,10,13], the question that remains to be answered is whether the same applies to natural earthquake faults. Interactions in a natural driven system should be mean field if the results on ergodicity are to hold. Since slider block models were originally conceived as models of earthquake faults, it is logical to investigate the presence or absence of ergodic behavior in systems of earthquake faults. We proceed to test this hypothesis using the TM fluctuation metric for seismicity.

Earthquakes in a natural fault system.—We apply the TM metric to the surface expression of the energy release in a regional fault network, the seismicity in central and southern California. The seismicity data employed in our analysis is taken from existing observations in California between the years 1932 and 2001. We compute the TM metric for seismicity over the region 32° to 40° latitude, $-115°$ to $-125°$ longitude. We use only events of magnitude $M \geq 3$ to ensure completeness of the catalog.

We combine the Southern California Earthquake Center (SCEC) database and the Northern California Seismic Network (NCSN) database [21]. The SCEC catalog is used for the region between 32° and 36° latitude, and the NCSN catalog between 36° and 40°. We bin the region into smaller boxes of 0.1° to a side and count the number of events in each box over time periods of one year in order to calculate $E_i(t) \equiv R_i(t)$. We have applied the same calculation to both larger and smaller box sizes, and the results discussed below hold, again supporting the conclusion that the results are a function of the long-range interactions in the system, not an effect of spatial coarse graining.

In Fig. 3 we plot the inverse TM metric over time for the number of events in California. Here, too, we observe the linear relationship between the inverse TM metric and time. Note the remarkable similarity to the slider block model shown in Fig. 2, even to the transient phase at the beginning. The natural fault system is effectively ergodic for relatively long periods of time, on the order of decades, punctuated by the occurrence of large earthquakes, such as the Kern County event of 1952, the Imperial Valley earthquake in 1979, and the Landers sequence of 1992. Between events the fault system resides in an ergodic, local energy minimum on a complex landscape. Eventually the nonlinear dynamics lead to an earthquake and the system migrates to a new local minimum, where it again resides in an effectively ergodic state.

In Fig. 4, top, we plot the temporal variance, the variance in the number of events at each cumulative moment magnitude level over the entire time period, using the same data and box size described above. Figure 4, bottom, shows the spatial variance, the variance in the number of events at the same cumulative magnitude levels over the entire spatial region. Each of these quantities approaches a constant for cumulative magnitude greater than 3.0, further evidence that the total number of events

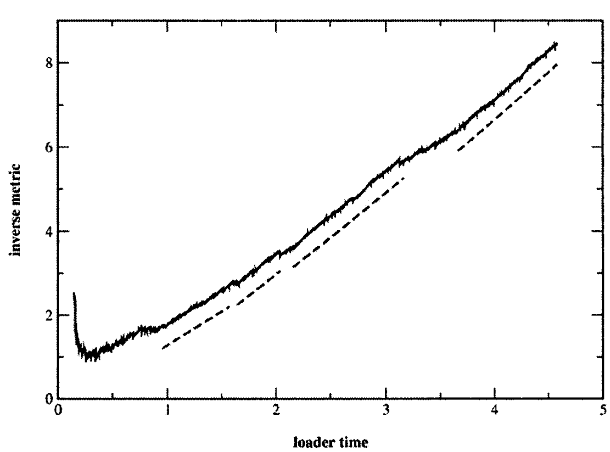

FIG. 2. The inverse TM metric plotted versus loader time, in a slider block model with precursory slip.

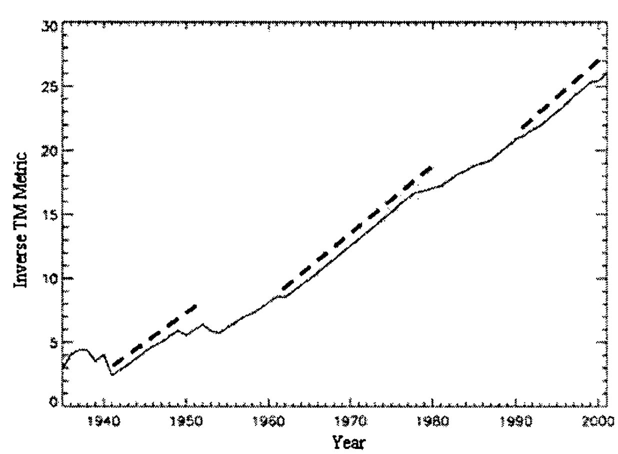

FIG. 3. The inverse TM metric for California seismicity.

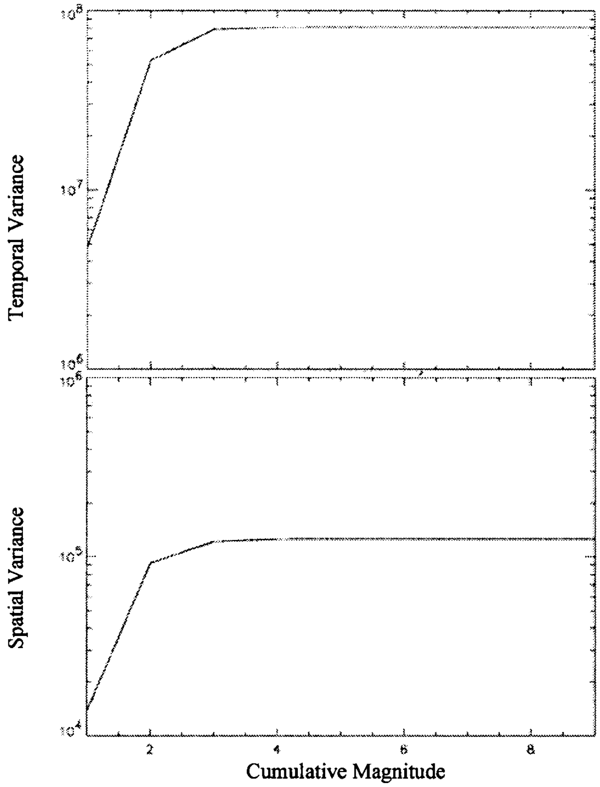

FIG. 4. The spatial and the temporal variance for cumulative magnitude in central and southern California.

is a constant over large enough spatial and temporal regions. The system is stationary for long time periods, as implied by the ergodicity constraint. Future studies will examine the relationship between spatial and temporal regions of varying sizes, their statistical properties, and the resulting ergodic behavior.

In conclusion, we employ here the Thirumalai-Mountain fluctuation metric to investigate one equilibrium property, effective ergodicity, in the dynamics of a natural threshold system, the earthquake fault system in California. Numerical simulations have demonstrated that the dynamics of driven mean-field systems of interacting slider block models and coupled map lattices display strong evidence of ergodic behavior. Our results suggest that this particular natural system is effectively ergodic and is mean field as in the numerical simulations that are used to study these systems, displaying critical point behavior with correlations over a range of spatial and temporal scales. It resides in metastable wells for significant periods of time. As the dynamical system evolves, it may suddenly jump to a new energy minimum with the occurrence of a large earthquake. This conclusion also suggests that many of the observed properties of the natural system, such as scaling, large correlation lengths, and the classification of earthquakes as nonclassical nucleation events, are manifestations of an effectively ergodic, nonlinear threshold system. The TM metric argues for the interpretation of natural earthquake systems as diffusive systems and may provide a means to better classify natural threshold systems in the future.

Research by K. F. T. was funded by the Southern California Earthquake Center, Contribution No. 705. SCEC is funded by NSF Cooperative Agreement No. EAR-8920136 and USGS Cooperative Agreements No. 14-08-0001-A0899 and No. 1434-HQ-97AG01718. Research by J. B. R. was funded by USDOE/OBES Grant No. DE-FG03-95ER14499 and by NASA Grant No. NAG5-5168. Research by W. K. and C. F. was supported by USDOE/OBES Grants No. DE-FG02-95ER14498 and No. W-7405-ENG-6 at LANL.

*Present address: University of Western Ontario, London, Ontario, Canada.

[1] J. B. Rundle *et al.*, Phys. Rev. Lett. **75**, 1658 (1995).
[2] D. Fisher *et al.*, Phys. Rev. Lett. **78**, 4885 (1997).
[3] A. V. M. Herz and J. J. Hopfield, Phys. Rev. Lett. **75**, 1222 (1995).
[4] D. S. Fisher, Phys. Rev. B **31**, 7233 (1985).
[5] J. S. Urbach, R. C. Madison, and J. T. Markert, Phys. Rev. Lett. **75**, 276 (1995).
[6] P. Bak, C. Tang, and K. Weisenfield, Phys. Rev. Lett. **59**, 381 (1987).
[7] A. D. Gopal and D. J. Durian, Phys. Rev. Lett. **75**, 2610 (1995).
[8] M. Kac, G. E. Uhlenbeck, and P. C. Hemmer, J. Math. Phys. (N.Y.) **4**, 216 (1963).
[9] I. G. Main, G. O'Brien, and J. R. Henderson, J. Geophys. Res. **105**, 6105 (2000).
[10] C. D. Ferguson, W. Klein, and J. B. Rundle, Phys. Rev. E **60**, 1359 (1999).
[11] D. Thirumalai and R. D. Mountain, Phys. Rev. E **47**, 479 (1993).
[12] D. Egolf, Science **287**, 101 (2000).
[13] W. Klein, J. B. Rundle, and C. D. Ferguson, Phys. Rev. Lett. **78**, 3793 (1997).
[14] J. L. Leibowitz, in *Statistical Mechanics: Fluctuation Phenomena*, edited by O. Penrose (North-Holland, Amsterdam, 1979), p. 295.
[15] R. F. Smalley, D. L. Turcotte, and S. A. Solla, J. Geophys. Res. **90**, 1894 (1985).
[16] Y. Huang *et al.*, Europhys. Lett. **41**, 43 (1998).
[17] H. F. Nijhout, in *Pattern Formation in the Physical and Biological Sciences*, Santa Fe Institute Lecture Notes V (Addison-Wesley, Reading, MA, 1997), p. 269.
[18] H. Kanamori, in *Earthquake Prediction: An International Review*, edited by D.W. Simpson II and P. G. Richards (AGU, Washington, D.C., 1981), p. 1.
[19] D. Hill, J. P. Eaton, and L. M. Jones, *USGS Professional Paper 1515* (U.S. GPO, Washington, D.C., 1990), p. 115.
[20] R. G. Palmer, Adv. Phys. **31**, 669 (1982).
[21] The SCEC database is available at www.scecdc.scec.org; the NCSN database is available at quake.geo.berkeley.edu.